

Group theoretical aspects of $L^2(\mathbb{R}^+)$, $L^2(\mathbb{R}^2)$ and associated Laguerre polynomials¹

E. Celeghini^{1,2}, M.A. del Olmo².

¹*Dipartimento di Fisica, Università di Firenze and INFN–Sezione di Firenze
150019 Sesto Fiorentino, Firenze, Italy*

²*Departamento de Física Teórica and IMUVA, Universidad de Valladolid,
E-47011, Valladolid, Spain.*

e-mail: celeghini@fi.infn.it, olmo@fta.uva.es

Abstract

A ladder algebraic structure for $L^2(\mathbb{R}^+)$ which closes the Lie algebra $h(1) \oplus h(1)$, where $h(1)$ is the Heisenberg-Weyl algebra, is presented in terms of a basis of associated Laguerre polynomials. Using the Schwinger method the quadratic generators that span the alternative Lie algebras $so(3)$, $so(2,1)$ and $so(3,2)$ are also constructed. These families of (pseudo) orthogonal algebras also allow to obtain unitary irreducible representations in $L^2(\mathbb{R}^2)$ similar to those of the spherical harmonics.

1 Introduction

The associated Laguerre polynomials (ALP) [1], $L_n^{(\alpha)}(x)$ ($x \in [0, \infty)$, $n = 0, 1, 2, \dots$ and α real fixed parameter, continuous and > -1), are defined by the 2nd order differential equation (DE)

$$\left[x \frac{d^2}{dx^2} + (1 + \alpha - x) \frac{d}{dx} + n \right] L_n^{(\alpha)}(x) = 0. \quad (1)$$

The ALPs reduce to the Laguerre polynomials for $\alpha = 0$. From the many recurrence relations that they verify [1, 2, 3], we start from the following ones

$$\left[-\frac{d}{dx} + 1 \right] L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x), \quad \left[x \frac{d}{dx} + \alpha \right] L_n^{(\alpha)}(x) = (n + \alpha) L_n^{(\alpha-1)}(x). \quad (2)$$

For $\alpha > -1$ and fixed, the ALP $L_n^{(\alpha)}(r)$ are orthogonal in the label n with respect the weight measure $d\mu(x) = x^\alpha e^{-x} dx$

$$\int_0^\infty dx x^\alpha e^{-x} L_n^{(\alpha)}(x) L_{n'}^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{nn'}.$$

¹Contribution to the 31st International Colloquium on Group Theoretical Methods in Physics, Rio de Janeiro, June 19-25, 2016. Accepted in *Springer Proceeding Series*.

For α integer such that $0 \leq \alpha \leq n$, we have the generalization [1]

$$L_n^{(-\alpha)}(x) := \frac{\Gamma(n - \alpha + 1)}{\Gamma(n + 1)} (-x)^\alpha L_{n-\alpha}^{(\alpha)}(x).$$

Hereafter we assume here $n \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, $n - \alpha \in \mathbb{N}$ and we consider α as a label, like n , and not a parameter fixed at the beginning.

Following the approach of previous works [4, 5, 6, 7] we introduce now a set of alternative functions including also the weight measure, in such a way to obtain the orthonormal bases we are used to in Quantum Mechanics

$$M_n^{(\alpha)}(x) := \sqrt{\frac{\Gamma(n + 1)}{\Gamma(n + \alpha + 1)}} x^{\alpha/2} e^{-x/2} L_n^{(\alpha)}(x).$$

For each fixed value of $\alpha \geq -n$ and $n \in \mathbb{N}$, the set of $M_n^{(\alpha)}(x)$, is a basis of $L^2(\mathbb{R}^+)$

$$\int_0^\infty M_n^{(\alpha)}(x) M_m^{(\alpha)}(x) dx = \delta_{nm}, \quad \sum_{n=0}^\infty M_n^{(\alpha)}(x) M_n^{(\alpha)}(x') = \delta(x - x').$$

2 The symmetry algebra $h(1)_n \oplus h(1)_p$

The eqs. (2) rewritten in terms of $M_n^{(\alpha)}$ take the form

$$\begin{aligned} \left[-\sqrt{x} \frac{d}{dx} + \frac{1}{2\sqrt{x}}(\alpha + x) \right] M_n^{(\alpha)}(x) &= \sqrt{n + \alpha + 1} M_n^{(\alpha+1)}(x), \\ \left[\sqrt{x} \frac{d}{dx} + \frac{1}{2\sqrt{x}}(\alpha + x) \right] M_n^{(\alpha)}(x) &= \sqrt{n + \alpha} M_n^{(\alpha-1)}(x), \end{aligned} \quad (3)$$

where $p := n + \alpha$ plays, for n fixed, the role of eigenvalue of the number operator in a Heisenberg-Weyl algebra, $h(1)$, realized on the space of functions $M_n^{(\alpha)}(x)$. It is indeed a positive integer like n , so that we can define the new functions $\mathcal{M}_{n,p}(x) := M_n^{(p-n)}(x)$, that by inspection are symmetric in the interchange $n \Leftrightarrow p$, i.e. $\mathcal{M}_{n,p}(x) = (-1)^{p-n} \mathcal{M}_{p,n}(x)$. The previous recurrence relations (3) can thus be rewritten

$$\begin{aligned} \left[-\sqrt{x} \frac{d}{dx} + \frac{\sqrt{x}}{2} + \frac{p-n}{2\sqrt{x}} \right] \mathcal{M}_{n,p}(x) &= \sqrt{p+1} \mathcal{M}_{n,p+1}(x), \\ \left[\sqrt{x} \frac{d}{dx} + \frac{\sqrt{x}}{2} + \frac{p-n}{2\sqrt{x}} \right] \mathcal{M}_{n,p}(x) &= \sqrt{p} \mathcal{M}_{n,p-1}(x). \end{aligned} \quad (4)$$

To construct the operatorial structure corresponding to the recurrence relations we define now four operators X, D_x, N and P

$$\begin{aligned} X \mathcal{M}_{n,p}(x) &= x \mathcal{M}_{n,p}(x), & D_x \mathcal{M}_{n,p}(x) &= \frac{d \mathcal{M}_{n,p}(x)}{dx}, \\ N \mathcal{M}_{n,p}(x) &= n \mathcal{M}_{n,p}(x), & P \mathcal{M}_{n,p}(x) &= p \mathcal{M}_{n,p}(x). \end{aligned}$$

Then, the 2nd order DE (1) becomes

$$\mathbb{E} \mathcal{M}_{n,p}(x) = 0, \quad (5)$$

where

$$\mathbb{E} := XD_x^2 + D_x + \frac{N+P+1}{2} - \frac{1}{4X}(P-N)^2 - \frac{X}{4}.$$

Moreover from (4) we get the differential operators (DOs)

$$\mathbf{b}^\pm := \mp \sqrt{X} D_x + \frac{\sqrt{X}}{2} + \frac{1}{2\sqrt{X}}(P-N), \quad (6)$$

that act on the functions $\mathcal{M}_{n,p}(x)$ in such a way that $\Delta n = 0$ and $\Delta p = \pm 1$. Since $[\mathbf{b}^-, \mathbf{b}^+] = \mathbb{I}$ they close an $h(1)$ algebra, $(h(1)_p)$ with quadratic Casimir $\mathcal{C}_p = \{\mathbf{b}^-, \mathbf{b}^+\} - 2(P+1/2)$ verifying $\mathcal{C}_p \mathcal{M}_{n,p}(x) = -2\mathbb{E} \mathcal{M}_{n,p}(x) = 0$.

Now taking into account the symmetry under the interchange $n \Leftrightarrow p$ of $\mathcal{M}_{n,p}(x)$ we can define the operators $\mathbf{a}^\pm(N, P) := -\mathbf{b}^\pm(P, N)$ that change the labels of $\mathcal{M}_{n,p}(x)$ as $\Delta p = 0$ and $\Delta n = \pm 1$. Their explicit action on $\mathcal{M}_{n,p}(x)$ is indeed

$$\mathbf{a}^+ \mathcal{M}_{n,p}(x) = \sqrt{n+1} \mathcal{M}_{n+1,p}(x), \quad \mathbf{a}^- \mathcal{M}_{n,p}(x) = \sqrt{n} \mathcal{M}_{n-1,p}(x).$$

The two operators \mathbf{a}^\pm determine thus another HW algebra, $h(1)_n$. Since these bosonic operators \mathbf{a}^\pm and \mathbf{b}^\pm commute among them we have obtained in this way the global algebra $h(1)_n \oplus h(1)_p$.

Moreover inside the Universal Enveloping Algebra $UEA[h(1)_n \oplus h(1)_p]$ other algebras preserving the parity of $n+p$ can be found by the Schwinger procedure [8] as we will do in the next section.

3 $so(3)$, $so(2, 1)$ and $so(3, 2)$ symmetries

$so(3)$ symmetry

We start from $J_\pm := \mathbf{a}_\pm \mathbf{b}_\mp$ obtaining 2nd order DOs that, taking into account eq. (5), can be rewritten in the space $\{\mathcal{M}_{n,p}(x)\}$ as 1st order DOs

$$J_\pm = \mp D_x (N - P \pm 1) + \frac{1}{2X} (N - P \pm 1)(N - P) - \frac{1}{2}(N + P + 1). \quad (7)$$

Defining $J_3 := (\mathbf{a}_- \mathbf{a}_+ - \mathbf{b}_- \mathbf{b}_+)/2 \equiv (N - P)/2$ we see that $\{J_\pm, J_3\}$ close a $su(2)$ algebra in the space $\{\mathcal{M}_{n,p}(x)\}$ since $[J_+, J_-] = 2J_3 - \frac{8}{X} J_3 \mathbb{E}$. The action of J_\pm is

$$J_+ \mathcal{M}_{n,p}(x) = \sqrt{(n+1)p} \mathcal{M}_{n+1,p-1}(x), \quad J_- \mathcal{M}_{n,p}(x) = \sqrt{n(p+1)} \mathcal{M}_{n-1,p+1}(x).$$

Also the Casimir of $su(2)$, $\mathcal{C}_{su(2)} = J_3^2 + \frac{1}{2}\{J_+, J_-\}$ is closely related to eq. (5) as $\mathcal{C}_{su(2)} = J(J+1) + \frac{1}{X}(4J_3^2 + 1)\mathbb{E}$, where J is the diagonal operator $J := (N+P)/2$.

$so(2, 1)$ symmetry

In a similar way we can define the operators $K_\pm := \mathbf{a}_\pm \mathbf{b}_\pm$, such that, like in the case of the operators J_\pm , we find in the space $\{\mathcal{M}_{n,p}(x)\}$

$$K_+ = XD_x + \frac{1}{2}(N+P+2-X), \quad K_- = -XD_x + \frac{1}{2}(N+P-X). \quad (8)$$

Both operators together with $K_3 := (\mathbf{a}_- \mathbf{a}_+ + \mathbf{b}_+ \mathbf{b}_-)/2 \equiv (N+P+1)/2$ determine a $su(1,1)$ algebra

$$[K_3, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_3,$$

since the action on the functions $\mathcal{M}_{n,p}(x)$ is

$$K_+ \mathcal{M}_{n,p}(x) = \sqrt{(n+1)(p+1)} \mathcal{M}_{n+1,p+1}(x), \quad K_- \mathcal{M}_{n,p}(x) = \sqrt{np} \mathcal{M}_{n-1,p-1}(x).$$

The Casimir of $su(1,1)$, $\mathcal{C}_{su(1,1)} = K_3^2 - \frac{1}{2}\{K_+, K_-\}$, is also connected with eq. (5) as $\mathcal{C}_{su(1,1)} = (M^2 - \frac{1}{4}) + X\mathbb{E}$, where $M = J_3 := (N-P)/2$.

More $so(2,1)$ symmetries

The commutators of J_\pm and K_\pm give the new operators

$$R_\pm := \pm[J_\pm, K_\pm], \quad S_\pm := \pm[J_\mp, K_\pm].$$

Provided that we define $R_3 := J + M + 1/2$ and $S_3 := J - M + 1/2$, they close two $so(2,1)$ algebras with commutators

$$[R_+, R_-] = -4R_3, \quad [R_3, R_\pm] = \pm 2R_\pm,$$

and Casimir $\mathcal{C}_R = R_3^2 - \frac{1}{2}\{R_+, R_-\} = -\frac{3}{4} + \frac{1}{X}(1 + (X+2M)^2)\mathbb{E}$ and similarly for $\{S_\pm, S_3\}$. Note that under the interchange $m \leftrightarrow -m$ we have $\{R_\pm, R_3\} \leftrightarrow \{S_\pm, S_3\}$.

$so(3,2)$ symmetry

All the operators $\{K_\pm, L_\pm, R_\pm, S_\pm, J, M\}$ can be written on the space $\{\mathcal{M}_{n,p}(x)\}$ as 1st order DOs. All together they determine on $\{\mathcal{M}_{n,p}(x)\}$ the representation of the Lie algebra $so(3,2)$ with $C_2^{so(3,2)} = -5/4$.

4 Representations of $so(3)$, $so(2,1)$ and $so(3,2)$ on the plane

We introduce now the operators directly related to $so(3)$, $J := (N+P)/2$ and $J_3 \equiv M := (N-P)/2$, and define

$$\mathcal{L}_j^m(x) := \mathcal{M}_{j+m,j-m}(x) = \sqrt{\frac{(j+m)!}{(j-m)!}} x^{-m} e^{-x/2} L_{j+m}^{(-2m)}(x).$$

The operators J_3 and J_\pm (7), rewritten in terms of J and M , act on $\{\mathcal{L}_j^m(x)\}$ as

$$J_3 \mathcal{L}_j^m(x) = m \mathcal{L}_j^m(x), \quad J_\pm \mathcal{L}_j^m(x) = \sqrt{(j \mp m)(j \pm m + 1)} \mathcal{L}_j^{m \pm 1}(x).$$

So, $\{\mathcal{L}_j^m(x)\}$ with $j \in \mathbb{N}$ and $|m| \leq j$ supports the representation \mathcal{D}_j of $so(3)$.

Similar results can be obtained for the other algebras $so(2,1)$ and $so(3,2)$. For instance, for the $so(2,1)$ spanned by $\{K_\pm, K_3\}$, $\{\mathcal{L}_j^m(x)\}$ supports the irreducible representation of the discrete series with Casimir $\mathcal{C}_{su(1,1)} := m^2 - \frac{1}{4}$ with m fixed and $j \geq |m|$.

On the other hand, in general these representations are not faithful because $\mathcal{L}_j^m(x) = \mathcal{L}_j^{-m}(x)$. The same difficulty is also present in the spherical harmonic where the associated Legendre polynomial P_l^m is related to P_l^{-m} . There the degeneration was removed by introducing an angle variable. Here we follow the same procedure by considering the new functions

$$\mathcal{Z}_j^m(r, \phi) := e^{im\phi} \mathcal{L}_j^m(r^2), \quad \phi \in \mathbb{R}, -\pi \leq \phi < \pi.$$

Under the change of variable $x \rightarrow r^2$ the DE (5) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{4m^2}{r^2} - r^2 + 4(j + \frac{1}{2}) \right] \mathcal{Z}_j^m(r, \phi) = 0.$$

Normalization and orthogonality of the $\mathcal{Z}_j^m(r, \phi)$ are similar to the ones of $Y_j^m(\theta, \phi)$

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \int_0^{\infty} 2r dr \mathcal{Z}_j^m(r, \phi)^* \mathcal{Z}_{j'}^{m'}(r, \phi) &= \delta_{j,j'} \delta_{m,m'}, \\ \sum_{j,m} \mathcal{Z}_j^m(r, \phi)^* \mathcal{Z}_j^m(r', \phi') &= \frac{\pi}{r} \delta(r-r') \delta(\phi-\phi'). \end{aligned}$$

This means that the set $\{\mathcal{Z}_j^m(r, \phi)\}$ is a basis in the space of square integrable functions defined on the plane, $L^2(\mathbb{R}^2)$, like $\{Y_j^m(\Omega)\}$ is a basis of $L^2(\mathbb{S}^2)$.

Moreover, with a convenient introduction of phases we can define the operators $\mathbb{J}_{\pm} := e^{\pm i\phi} J_{\pm}$ and $\mathbb{J}_3 := J_3$, in the finite dimensional space $\{\mathcal{Z}_j^m(r, \phi)\}$ with fixed j

$$\mathbb{J}_{\pm} \mathcal{Z}_j^m(r, \phi) = \sqrt{(j \mp m)(j \pm m + 1)} \mathcal{Z}_j^{m \pm 1}(r, \phi), \quad \mathbb{J}_3 \mathcal{Z}_j^m(r, \phi) = m \mathcal{Z}_j^m(r, \phi),$$

and analogously for the remaining operators. So $\{\mathcal{Z}_j^m(r, \phi)\}$ support irreducible representations of $so(3)$, $so(2, 1)$ and $so(3, 2)$ on the plane as $\{Y_j^m(\theta, \phi)\}$ are on the sphere. For more details see [7, 9, 10].

From the physical point of view, in spite of the analogy with the angular momentum, \mathbb{J}_{\pm} and \mathbb{J}_3 can be related to a one-dimensional Morse system, where m and j are connected with the potential [9].

Conclusions

A relationship between Lie algebras and square integrable functions has been found. Indeed we need to restrict ourselves to $L^2(\mathbb{R}^+)$ and $L^2(\mathbb{R}^2)$, where \mathbb{E} is identically zero, to obtain differential representations of Lie algebras in the spaces of functions defined in \mathbb{R}^+ and \mathbb{R}^2 .

Acknowledgements

This work was partially supported by the Ministerio de Economía y Competitividad of Spain (Project MTM2014-57129-C2-1-P with EU-FEDER support).

References

- [1] G. Szegő, *Orthogonal Polynomials*, (Am. Math. Soc., Providence, 2003), pp. 100-105
- [2] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark, *NIST Handbook of Mathematical Functions*, (Cambridge Univ. Press, New York, 2010)
- [3] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1972)
- [4] E. Celeghini, M.A. del Olmo, *Ann. Phys.* **335** (2013) 78-85
- [5] E. Celeghini, M.A. del Olmo, *Ann. Phys.* **333** (2013) 90-103
- [6] E. Celeghini, M.A. del Olmo, M.A. Velasco, *J. Phys.: Conf. Ser.* **597** (2015) 012023
- [7] E. Celeghini, M.A. del Olmo, arXiv: 1504.01572 [math-ph]
- [8] J. Schwinger, in *Quantum Theory of Angular Momentum* (L. Biedenharn, E. van Dam, Eds.), (Academic Press, New York, 1965), pp. 229-279
- [9] Y. Alhassid, F. Gürsey, F. Iachello, *Ann. Phys.* **148** (1983) 346-380
- [10] J. Guerrero, V. Aldaya, *J. Phys. A* **39** (2006) L267-L276.